$$\cos(\text{arc}\tan q - \text{arc}\tan q^3 + \text{arc}\tan q^5 - \text{\&c.})$$
$$= \frac{1-q^6-q^{10}+q^{28}+q^{36}-\text{\&c.}}{(1+q^2)^{\frac{1}{2}}(1+q^6)^{\frac{1}{2}}\ldots 1-q^4 . 1-q^8 \ldots} \quad . \quad . \quad . \quad (10),$$

$$\sin(\text{arc}\tan q - \text{arc}\tan q^3 + \text{arc}\tan q^5 - \text{\&c.})$$
$$= \frac{q-q^3-q^{15}+q^{21}+q^{45}-\text{\&c.}}{(1+q^2)^{\frac{1}{2}}(1+q^6)^{\frac{1}{2}}\ldots 1-q^4 . 1-q^8 \ldots} \quad . \quad . \quad . \quad (11),$$

the exponents being the triangular numbers. These formulæ can be readily identified with (8) and (9); for

$$(1+q . 1+q^3 \ldots)^{\frac{1}{2}} = \frac{2^{\frac{1}{12}}q^{\frac{1}{48}}}{k^{\frac{1}{24}}k'^{\frac{1}{24}}},$$

$$(1-q^2 . 1-q^4 \ldots) = \frac{2^{\frac{1}{6}}\text{K}^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}k^{\frac{1}{6}}k'^{\frac{1}{6}}q^{-\frac{1}{12}},$$

so that, replacing q by $q^{\frac{1}{2}}$ and putting for

$$\text{arc}\tan q^{\frac{1}{2}} - \text{arc}\tan q^{\frac{3}{2}} + \text{arc}\tan q^{\frac{5}{2}} - \text{\&c.},$$

its value $\frac{1}{4}\theta$, (10) and (11) become

$$\cos\tfrac{1}{4}\theta = \frac{\pi^{\frac{1}{2}}}{2^{\frac{1}{4}}\text{K}^{\frac{1}{2}}}\frac{q^{\frac{1}{16}}}{k^{\frac{1}{8}}k'^{\frac{1}{8}}}(1-q^3-q^5+q^{14}+q^{18}-\text{\&c.}),$$

$$\sin\tfrac{1}{4}\theta = \frac{\pi^{\frac{1}{2}}}{2^{\frac{1}{4}}\text{K}^{\frac{1}{2}}}\frac{q^{\frac{1}{16}}}{k^{\frac{1}{8}}k'^{\frac{1}{8}}}(q^{\frac{1}{2}}-q^{\frac{3}{2}}-q^{\frac{15}{2}}+q^{\frac{21}{2}}+q^{\frac{45}{2}}-\text{\&c.}),$$

which are in fact (8) and (9).

X. "On certain Definite Integrals." No. 5. By W. H. L. Russell, F.R.S. Received October 13, 1879.

The following is a continuation of four papers inserted in the "Proceedings of the Royal Society." In my last paper I gave a proof that

$$\sin(2n+3)\theta - \sin(2n+1)\theta = 0$$

when (n) is infinite. We may consider the subject also thus. Taking the expression

$$\frac{(2n+3)\theta - \dfrac{(2n+3)^3}{1 . 2 . 3}\theta^3 + \ldots \pm \dfrac{(2n+3)^{2m+1}}{1 . 2 \ldots 2m+1}\theta^{2m+1}}{(2n+1)\theta - \dfrac{(2n+1)^3}{1 . 2 . 3}\theta^3 \quad \ldots \pm \dfrac{(2n+1)^{2m+1}}{1 . 2 . 3 . 2m+1}\theta^{2m+1}}$$

we see manifestly that as (n) increases to infinity the fraction becomes unity. And this is true however large (m) is considered to be.

Manifestly therefore when (n) is infinite $\frac{\sin(2n+3)\theta}{\sin(2n+1)\theta}=1$, another form of the same theorem.

Moreover $\alpha\sin\theta+\alpha^3\sin 3\theta+\ldots=\frac{(\alpha+\alpha^3)\sin\theta}{1-2\alpha^2\cos 2\theta+\alpha^4}$ when α is less than unity.

As α approaches unity in the limit,

$$\sin\theta+\sin 3\theta+\sin 5\theta\ldots=\frac{1}{2\sin\theta},$$

a well-known theorem.

But also

$$\sin\theta+\sin 3\theta+\ldots\sin(2n+1)\theta=\frac{2\sin\theta-\sin(2n+3)\theta+\sin(2n+1)\theta}{2(1-\cos 2\theta)}.$$

Making (n) infinite and subtracting the last equation we have again $\sin(2n+3)\theta-\sin(2n+1)\theta=0$.

I have considered this subject at length on account of a remark of Professor Stokes, that $\sin(2n+3)\theta-\sin(2n+1)\theta$ cannot converge to zero in the ordinary manner, on account of the perpetual fluctuation of the signs of $\sin(2n+3)\theta$ and $\sin(2n+1)\theta$. There can only I think be one explanation of this, namely, that the sine of an infinite angle is zero. Following up then the theorem we find

$$(86.)\quad \int_0^\pi \frac{d\theta}{\sin\theta}\epsilon^{\alpha\cos\theta}\sin(\alpha\sin\theta)=\frac{\pi}{2}(\epsilon^{\alpha}-\epsilon^{-\alpha}).$$

$$(87.)\quad \int_0^\pi \frac{d\theta}{\sin\theta}\cos\alpha\sin\theta(\epsilon^{\alpha\sin\theta}-\epsilon^{-\alpha\sin\theta})=2\pi\sin\alpha.$$

$$(88.)\quad \int_0^\pi \frac{d\theta\sin(2n+1)\theta}{\sin\theta}=\pi.$$

We shall also obtain the following:—

$$(89.)\quad \int_0^\pi d\theta\epsilon^{x\cos\theta}\cos(x\sin\theta+\theta)\log_\epsilon(1+2\alpha\cos\theta+\alpha^2)=\frac{\pi(\epsilon^{\alpha x}-1)}{x\epsilon^{\alpha x}},$$

$$(90.)\quad \int_0^\pi d\theta\,.\,\frac{\epsilon^{x\cos\theta}\cos(x\sin\theta-\theta)+\cos\theta}{\epsilon^{2x\cos\theta}+2\epsilon^{x\cos\theta}\cos(x\sin\theta)+1}\log_\epsilon(1-2\alpha\cos\theta+\alpha^2)$$
$$=\frac{\pi}{x}\log_\epsilon\frac{\epsilon^{\alpha x}+1}{2\epsilon^{\alpha x}}.$$

$$(91.)\quad \int_0^\pi d\theta\,.\,\frac{(1+\alpha^2)\cos\theta+2\alpha}{(1+2\alpha\cos\theta+\alpha^2)^2}\log_\epsilon(1+2\alpha\cos\theta+\alpha^2)=\frac{2\pi\alpha}{1-\alpha^2}.$$

Since $$\frac{x^2}{1\,.\,2}+\frac{x^3}{2\,.\,3}+\frac{x^4}{3\,.\,4}+\ldots=x+(1-x)\log_\epsilon(1-x),$$

when x is any quantity from 0 to 1, inclusive, we shall have

$$\frac{\cos 2\theta}{1\,.\,2}+\frac{\cos 3\theta}{2\,.\,3}+\frac{\cos 4\theta}{3\,.\,4}+\ldots=\cos\theta+2\sin^2\frac{\theta}{2}\log_\epsilon\left(2\sin\frac{\theta}{2}\right)-\frac{\pi-\theta}{2}\sin\theta.$$

We shall call this expression Θ, and shall write more generally

$$\Theta_{m,\,n}^{(r)}=\frac{\cos r\theta}{mn}+\frac{\cos(r+1)\theta}{(m+1)(n+1)}+\ldots,$$

then it is manifest that $\Theta_{2,\,3}^{(3)}$, $\Theta_{3,\,4}^{(4)}$, ... may be found immediately in terms of Θ.

Again, since

$$\frac{x^3}{1\,.\,3}+\frac{x^4}{2\,.\,4}+\frac{x^5}{3\,.\,5}+=\frac{x}{2}+\frac{x^2}{4}+\frac{1-x^2}{2}\log_\epsilon(1-x),$$

we may find a precisely similar expression for $\Theta_{1,\,3}^{(3)}$ and consequently for $\Theta_{2,\,4}^{(4)}$, $\Theta_{3,\,5}^{(5)}$ . . .

We thus obtain

(92.) $$\int_0^\pi \Theta\cos r\theta\,.\,d\theta=\frac{\pi}{2(r+1)r}.$$

(93.) $$\int_0^\pi \Theta d\theta\,.\,\frac{\alpha^2\cos 2\theta-\alpha^3\cos\theta}{1-2\alpha\cos\theta+\alpha^2}=\frac{\pi}{2}\{\alpha+(1-\alpha)\log_\epsilon(1-\alpha)\}.$$

(94.) $$\int_0^\pi \Theta d\theta\epsilon^{\alpha\cos\theta}\cos(\alpha\sin\theta+2\theta)=\frac{\pi}{2}\left\{\frac{\epsilon^\alpha}{\alpha^2}-\frac{1+\alpha}{\alpha^2}\right\}.$$

(95.) $$\int_0^\pi \Theta_{12}^{(3)}d\theta\{\log_\epsilon(1+2x\cos\theta+x^2)+2x\cos\theta+x^2\cos 2\theta\}$$
$$=\frac{\pi}{2}\{(1-x)^2\log_\epsilon(1-x)-(x^2+x)\}.$$

(96.) $$\int_0^\pi d\theta\Theta_{23}^{(3)}\epsilon^{\alpha\cos\theta}\cos(\alpha\sin\theta+3\theta)=\frac{\pi}{2\alpha^3}\{\epsilon^\alpha(\alpha-2)+(\alpha+2)\}.$$

(97.) $$\int_0^\pi d\theta\Theta_{2,\,4}^{(4)}\epsilon^{\alpha\cos\theta}\cos(\alpha\sin\theta+4\theta)=\frac{\pi}{2\alpha^4}\left\{\epsilon^\alpha(\alpha^2-3\alpha+3)+\frac{\alpha^2}{2}-3\right\}.$$

(98.) $$\int_0^\pi d\theta\,.\,\Theta_{12}^{(4)}\Theta_{34}^{(4)}=\frac{\pi}{36}.$$ (99.) $$\int_0^\pi d\theta\Theta_{12}^{(5)}\Theta_{45}^{5}=\frac{5\pi}{288}.$$

More generally we may find

(100.) $$\int_0^\pi d\theta\Theta_{m,\,n}^{(s)}\Theta_{r,\,s}^{(s)},$$

where m, n, r, s, are in order of magnitude.

The instance I have mentioned at the commencement of this paper, is not the only instance of my obligation to the kindness of Professor Stokes. He prevented me on another occasion from being led to a false conclusion, by taking for granted without due examination an integral which had appeared in print.